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BRIEF COMMUNICATION

INTERFACIAL BOUNDARY CONDITIONS ON A DROPLET

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1. INTRODUCTION

The solution of droplet or bubble motion in Stokes flows is very similar to problems concerning rigid particles. The governing field equations in both cases are the same, yet the boundary conditions imposed on the interface of a drop are more complex and couple the outer field region with the fluid field interior to the drop. Thus, the flow field exterior to the drop has to be solved simultaneously with the flow field interior to it. The no-slip condition on the boundary of a solid particle suspended in Stokes flows, implies that three components of the velocity field and the pressure field have to be obtained, subject to three boundary conditions on the surface of the particle, in addition to the boundary condition at infinity.

For a similar problem concerning a drop, three components of the velocity field exterior to the drop and three components of the velocity field interior to the drop have to be calculated subject to seven boundary conditions to be satisfied on the interface.

The seven boundary conditions employed on the interface (assuming absence of surfactants) are as follows:

(a) The continuity of the velocity field through the interface yields three scalar boundary conditions.

(b) No mass transfer through the interface yields one boundary condition.

(c) Continuity of the tangential components of the stress vector acting on the interface yields two boundary conditions.

(d) The last boundary condition links the normal component of the stress vector acting on the interface and the surface tension.

For a *spherical* drop where the inequality $(\mu|\underline{s}|a/\sigma) \ll 1$ holds, (μ, $|\underline{s}|$ denote the viscosity and the strain rate of the outer field respectively, a is the droplet's radius, and σ is the surface tension) the last boundary condition is superfluous (Hetsroni & Haber 1970). For a problem including n spherical drops suspended in a Stokesian field, one should have to satisfy $6n$ boundary conditions compared to $3n$ boundary conditions of the equivalent case concerning rigid particles.

This note presents a new formulation for the existing boundary conditions imposed on the interface of a spherical drop. The suggested presentation decouples the flow field interior to the drop from the one exterior to it and provides a set of equations which is sufficient to determine the flow field exterior to the drop and the drag force acting on the drop without the necessity of solving simultaneously the flow field interior to the drop.

2. THE SOLUTION

The boundary conditions, employed on the interface of a droplet, couple the velocities and

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the stresses of the flow fields interior to the drop and exterior to it, i.e.

$$\left.\begin{aligned} \mathbf{v}\cdot\hat{\mathbf{r}} &= \mathbf{U}\cdot\hat{\mathbf{r}} \qquad [1]\\ \mathbf{v} &= \mathbf{u} \qquad [2]\\ \hat{\mathbf{r}}\cdot\underline{\Pi}\cdot(\underline{\mathbf{I}}-\hat{\mathbf{r}}\hat{\mathbf{r}}) &= \hat{\mathbf{r}}\cdot\underline{\tau}\cdot(\underline{\mathbf{I}}-\hat{\mathbf{r}}\hat{\mathbf{r}}) \qquad [3] \end{aligned}\right\} r = a,$$

where $\mathbf{v}$ and $\underline{\Pi}$ denote the velocity and stress fields exterior to the drop (outer fields), $\mathbf{u}$ and $\underline{\tau}$ denote the velocity and stress fields interior to the drop (inner fields), $\mathbf{U}$ is the terminal velocity of the drop, $\underline{\mathbf{I}}$ is the idem tensor and $\hat{\mathbf{r}}$ is a unit vector normal to the interface.

These conditions are homogeneous, except boundary condition [1] satisfied by the component of the velocity field normal to the interface (which depends on the terminal velocity of the drop).

The components of the velocity and stress vector fields tangential to the interface are unknown in advance. However, their evaluation at the interface should yield some regular functions of θ and ϕ (where r, θ, ϕ are spherical coordinates with the origin located at the center of the drop). The non-singularity of these functions stems from the fact that both the velocity and the stresses are finite. Moreover, since the velocity field satisfies Stokes equations, it must be differentiable twice while the stress should be at least differentiable once.

The velocity field interior to a spherical shape drop can be derived, provided the velocity field is a known prescribed function at the interface. Lamb's general solution (Happel & Brenner 1965) may furnish the desired velocity and pressure fields, i.e.

$$\mathbf{u} = \sum_{n=1}^{\infty}\left\{\nabla\times(\mathbf{r}h_n) + \nabla r_n + \frac{n+3}{2(n+1)(2n+3)}\nabla(r^2 q_n) - \frac{n}{(n+1)(2n+3)}\mathbf{r}q_n\right\}, \qquad [4]$$

$$q = \mu_i \sum_{n=0}^{\infty} q_n,$$

where $\mathbf{r}$ is the position vector measured from the center of the drop, q and μ_i denote the pressure and viscosity of the inner field and r_n, h_n and q_n are solid spherical harmonics. Thus, the functional form of $\mathbf{u}$ and q is readily obtained while only the unknown coefficients of the solid harmonics have to be derived. These coefficients are determined utilizing the boundary conditions given at the interface of the drop.

Assuming that the solution of the outer field is known, then the boundary conditions [1] and [2], were sufficient to determine the inner field; namely all the unknown coefficients of the inner field can be determined via the *assumed* known velocity components. However, the boundary condition [3] relating the stress fields, has to be satisfied also. This boundary condition yields a redundant set of equations for the same unknown coefficients of the inner field. In order to have just one solution, certain relationships must exist between the velocity and the stress fields of the outer field evaluated at the interface. These relationships are the boundary conditions specified on the interface of the drop pertaining to the outer field, i.e.

$$\mathbf{v}\cdot\hat{\mathbf{r}} = \mathbf{U}\cdot\hat{\mathbf{r}}, \qquad [5]$$

$$\int_S \{\mu_i(n-1)[\hat{\mathbf{r}}\cdot\nabla\times\mathbf{v}] - \mathbf{r}\cdot\nabla\times\mathbf{\Pi}_{(r)}\} S_n^m(\theta,\phi)\,\mathrm{d}s = 0, \qquad [6]$$

$$\int_S \left\{\mu_i\left[(n-1)v_r + (2n+1)\left(r\frac{\partial v_r}{\partial r}\right)\right] + \mathbf{r}\cdot\nabla\times(\mathbf{r}\times\mathbf{\Pi}_{(r)})\right\} S_n^m(\theta,\phi)\,\mathrm{d}s = 0, \qquad [7]$$

where $v_r = \mathbf{v} \cdot \hat{\mathbf{r}}$, $\Pi_{(r)} = \underline{\Pi} \cdot \hat{\mathbf{r}}$ and ds denotes a surface element on the interface and $S_n^m(\theta, \phi)$ is a surface harmonic, namely:

$$S_n^m(\theta,\phi) = {\cos \atop \sin} m\phi P_n^m(\cos\theta) \quad {m = 0, 1, 2, \ldots, n \atop n = 1, 2, 3, \ldots, \infty}, \tag{8}$$

where $P_n^m(\cos\theta)$ denotes the associated Legendre polynomial.

The method used to derive [6] and [7] is outlined in the following paragraph.

3. METHOD OF SOLUTION

The boundary conditions [1]–[3] can be transformed as follows (Hetsroni & Haber 1970):

$$\left.\begin{aligned} \mathbf{v} \cdot \hat{\mathbf{r}} &= \mathbf{u} \cdot \hat{\mathbf{r}} \\ r\frac{\partial v_r}{\partial r} &= r\frac{\partial u_r}{\partial r} \\ \mathbf{r} \cdot \nabla \times \mathbf{v} &= \mathbf{r} \cdot \nabla \times \mathbf{u} \\ \mathbf{r} \cdot \nabla \times \Pi_{(r)} &= \mathbf{r} \cdot \nabla \times \tau_{(r)} \\ \mathbf{r} \cdot \nabla \times (\mathbf{r} \times \Pi_{(r)}) &= \mathbf{r} \cdot \nabla \times (\mathbf{r} \times \tau_{(r)}) \end{aligned}\right\} \text{ at } r = a. \tag{9}$$

Utilizing [4] one obtains the following equations

$$\sum_{n=1}^{\infty} \left\{ \frac{na}{2(2n+3)} q_n + \frac{n}{a} r_n \right\} = v_r \tag{10}$$

$$\sum_{n=1}^{\infty} \left\{ \frac{n(n+1)a}{2(2n+3)} q_n + \frac{n(n-1)}{a} r_n \right\} = r\frac{\partial v_r}{\partial r} \quad \text{at } r = a \tag{11}$$

$$\sum_{n=1}^{\infty} n(n+1) h_n = \mathbf{r} \cdot \nabla \times \mathbf{v} \tag{12}$$

$$-\mu_i \sum_{n=1}^{\infty} \left\{ \frac{2(n-1)(n+1)n}{a} r_n + \frac{n^2(n+2)}{2n+3} q_n \right\} = \mathbf{r} \cdot \nabla \times (\mathbf{r} \times \Pi_{(r)}) \quad \text{at } r = a. \tag{13}$$

$$\frac{\mu_i}{a} \sum_{n=1}^{\infty} n(n-1)(n+1) h_n = \mathbf{r} \cdot \nabla \times \Pi_{(r)}. \tag{14}$$

In general the solid harmonics q_n, r_n and h_n have the following form:

$$\begin{aligned} q_n &= r^n \sum_{m=0}^{n} (Q_n^m \cos m\phi + \hat{Q}_n^m \sin m\phi) P_n^m(\cos\theta), \\ r_n &= r^n \sum_{m=0}^{n} (R_n^m \cos m\phi + \hat{R}_n^m \sin m\phi) P_n^m(\cos\theta), \\ h_n &= r^n \sum_{m=0}^{n} (H_n^m \cos m\phi + \hat{H}_n^m \sin m\phi) P_n^m(\cos\theta), \end{aligned} \tag{15}$$

where $r = |\mathbf{r}|$, Q_n^m, $\hat{Q}_n^m$, etc. are unknown constant coefficients to be determined.

Substituting [15] in the l.h.s. of [10]–[14] using the orthogonal properties of the Legendre polynomials one can eliminate the coefficients Q_n, R_n, H_n, etc. and obtain [6] and [7]. It should be noted that no more than first order derivatives of the stress vector are included in this formulation. This consistent with the arguments at the beginning of section 2.

To represent [6] and [7] in invariant form (independent of the choice of the coordinate system, i.e. the angles θ and ϕ), one can utilize the n rank tensors $\mathbf{S}n$ defined by Brenner (1964) and replace $Sn(\theta, \phi)$ by $\mathbf{S}n$ where

$$\mathbf{S}n = \frac{(-1)^n}{n!} r^{n+1}\nabla^n \left(\frac{1}{r}\right), \qquad [16]$$

e.g.

$$\mathbf{S}_1 = \frac{\mathbf{r}}{r}, \quad \mathbf{S}_2 = \frac{1}{2}\left(3\frac{\mathbf{rr}}{r^2} - \underline{\mathbf{I}}\right), \quad \text{etc}.$$

Thus [6] and [7] are the boundary conditions applied to the outer velocity field v decoupled from the interior flow field **u**. They depend solely on μ_i and a which are the only parameters of the field interior to the drop.

4. EXAMPLE

Since the governing field equations for a droplet and a solid sphere are equivalent, the general solution should be the same, i.e.

$$\mathbf{v} = \left[\left(A_1\frac{a}{r} + A_2\frac{a^3}{r^3}\right)\underline{\mathbf{I}} + \left(A_3\frac{a}{r} + A_4\frac{a^3}{r^3}\right)\hat{\mathbf{r}}\hat{\mathbf{r}}\right] . \mathbf{U}, \qquad [17]$$

$$p = \mu_0 A_5 \frac{a}{r^2}\hat{\mathbf{r}} . \mathbf{U}, \qquad [18]$$

where A_1 and A_5 are unknown coefficients. In order to satisfy the continuity equation one readily obtains

$$A_1 = A_3, \qquad [19]$$

$$-3A_2 = A_4. \qquad [20]$$

The following functions of v should be calculated:

$$v_r = \left[2A_1\frac{a}{r} - 2A_2\left(\frac{a^3}{r^3}\right)\right][\hat{\mathbf{r}} . \mathbf{U}], \qquad [21]$$

$$\left[r\frac{\partial v_r}{\partial r}\right]_{r=a} = (-2A_1 + 6A_2)(\hat{\mathbf{r}} . \mathbf{U}), \qquad [22]$$

and

$$\frac{1}{\mu_0}\mathbf{\Pi}_{(r)} = \left[2A_4\frac{a^3}{r^4}\underline{\mathbf{I}} + \begin{pmatrix}\text{function of } r, a \text{ and the}\\ \text{unknown coefficients}\end{pmatrix}\hat{\mathbf{r}}\hat{\mathbf{r}}\right] . \mathbf{U}, \qquad [23]$$

where the term in brackets is immaterial for further calculations. Utilizing boundary condition [5] one obtains easily

$$2A_1 - 2A_2 = 1. \qquad [24]$$

Substituting [17] and [23] in boundary condition [6] yields equations which vanish identically.

Substituting [22] and [23] in boundary condition [7] yields:

$$[-6\lambda A_1 + 6(3\lambda + 2)A_2]\mathbf{U}_0 . \int_s \hat{\mathbf{r}}\hat{\mathbf{r}}\,\mathrm{d}s = \mathbf{0}, \qquad [25]$$

for $n = 1$. For $n > 1$ the equations vanish identically. Since the integral in [25] is $(4\pi/3)a^2\underline{\mathbf{I}}$ the term in brackets must vanish, i.e.

$$-\lambda A_1 + (2+3\lambda)A_2 = 0. \qquad [26]$$

Equations [19], [20], [24] and [26] are sufficient to determine the unknowns A_1, A_2, A_3 and A_4, i.e.

$$A_1 = A_3 = \frac{2+3\lambda}{1+\lambda}, \qquad [27]$$

$$A_2 = \frac{-A_4}{3} = \frac{\lambda}{4(1+\lambda)}. \qquad [28]$$

The pressure p can be determined by a straightforward substitution of [17] in the momentum equation.

5. CONCLUSIONS

Equations [6] and [7] provide a new formulation for the existing boundary conditions on the interface of a spherical drop. The solution of Hadamard (1911) and Rybczynski (1911) is easily obtained as shown in paragraph 4. The solution of two drops moving along their line of centers was obtained by Haber *et al.* (1974) by means of a stream function and is not repeated here because of its complexity. The problem concerning two drops moving perpendicular to their line of centers has not been solved in a closed form solution.

The method presented by O'Neill & Majumdar (1970) for the solution of two rigid spheres moving in the same configuration is not as applicable as one might think. Since the number of the undetermined coefficients increases rapidly (because of the two extra fields interior to the drops to be simultaneously determined) the solution relies heavily on numerical calculations. The boundary conditions represented in [6] and [7] may provide a new method for the solution of that problem (and other related problems) by analytical or numerical integration.

The numerical integration seems to be most promising, since the general solution presented by bi-spherical harmonics is determined by conditions on the boundaries of a finite rectangle. On the other hand, the solution which relies on the regular boundary conditions has to be determined by conditions imposed on the boundaries of an infinite band.

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